Elementary Derivation of the Dirac Equation. VI

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The transversality in the electrodynamic hydrogen field is discussed.

In [1] it has been shown that the system of differential equations of electrodynamics and that of relativistic wave mechanics are isomorphic:

$$\text{Rel. WM} \equiv \text{Electrodyn.} \tag{1}$$

This identity is not symmetrical since, according to [1], it indeed holds for the whole of wave mechanics, but only for that part of electrodynamics which is restricted by [1], (3). Electrodynamics is therefore the more comprehensive theory and contains wave mechanics as a special case. In the following we will denote by "Dirac-like electrodynamics" that part of electrodynamics which coincides with relativistic wave mechanics.

With regard to the solutions one can state: The set of all solutions of relativistic wave mechanics is a true subset of the set of all solutions of electrodynamics. And: Each solution of relativistic wave mechanics is also a solution of WM-isomorphic electrodynamics and vice versa.

In particular, the hydrogen solution of relativistic wave mechanics is also the hydrogen solution of Dirac-like electrodynamics. Thus we have at our disposal a wave field solution in electrodynamics which reproduces exactly the spectroscopic data of hydrogen.

With regard to the predictive content of the two isomorphic theories it is to be expected that the solutions of the six-component Dirac-like electrodynamics will contain much more information than the corresponding four-component solutions of relativistic wave mechanics. Dirac-like electrodynamics, according to [1], (4), has the form

$$\operatorname{rot} \boldsymbol{H} = \frac{\varepsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \quad \operatorname{rot} \boldsymbol{E} = -\frac{\mu}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \tag{2}$$
$$\operatorname{div} \boldsymbol{E} = 0, \quad \operatorname{div} \boldsymbol{H} = 0.$$

According to Ehrenfest's Theorem, Newton's mechanics is the center of gravity average of Schrödinger's wave mechanics:

$$\langle \text{WM} \rangle_{\text{Ehrenfest}} \equiv \text{Newton's mechanics}. \tag{3}$$

In analogy, relativistic mechanics has to be the center of gravity average of relativistic wave mechanics, or respectively, of Dirac-like electrodynamics:

$$\langle \text{rel. MW} \rangle_{\text{Ehrenfest}} \equiv \text{rel. mech.} \tag{4}$$

or, because of (1)

$$\langle \text{Dirac-like electrodyn} \rangle_{\text{Ehrenfest}} \equiv \text{rel. mech.} \tag{5}$$

The relation (4) has not been implemented yet computationally, and in (5) the right hand side could possibly be more general than relativistic mechanics because of the six components of the left hand side. Equations (1), (3), (4) and (5) make it, however, evident that mechanics and electrodynamics are identical theories – one is the average of the other.

In particular, the averaging procedure as applied to the wave mechanical hydrogen model leads to Bohr's hydrogen model, a mechanical two-body system in Keplerian motion. Since the scattering experiments of Rutherford the two bodies have been regarded as proton and electron, whose data are fitted to the spectroscopic hydrogen data through the Rydberg number and a quantized scheme for the orbit radii of the electron. For the ground state an orbit radius of the order of 10^{-8} cm is obtained. This is satisfactory if one may view the orbiting electron as a physical delimitation of the atom. In repetitions of Rutherford's scattering experiments, however, it was found again and again that the orbiting electrons did not manifest themselves in any way. They rather behaved as if they were not present at all.

With regard to the corresponding hydrogen model of Dirac-like electrodynamics the following questions arise:

1) What do the electromagnetic fields of the hydrogen model of Dirac-like electrodynamics look like?

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2) What should be the two bodies in the electrodynamics hydrogen model which move around each other in a Keplerian motion?

Ad 2): The third equation in (2) requires that the complete solution should be charge free. The two charge carriers proton and electron, which, since Rutherford, represent nucleus and orbiter of the mechanical, wave mechanical and relativistic wave mechanical hydrogen model, therefore are not suitable for the electrodynamic hydrogen model. According to the relevant considerations in [3], the two bodies are rather two photons, or respectively, two photon-wave fields, whose centers of gravity move around in a Kepler motion. This can be illustrated more specifically as follows: In view of Hamilton's analogy a particle in a potential behaves like a light ray in a refractive medium. Or: each problem in optics is equivalent to one in mechanics and vice versa.

In the special case of a two-body system the analogy states: A particle in a potential or in the field of another particle acts in the same way as a photon in a refractive medium or in the field of another photon.

Carrying the specialization further to potentials of inverse distance one can state: Two particles, if close together, impede each other in their straight inertial pathes interacting in such a way that they finally come into an everlasting Kepler motion. And with Hamilton's analogy: Two neighboring light rays (photons) impede each other in their straight trajectories interacting in such a way that they (or respectively, the centers of gravity of their wave fields) finally come into an ever-lasting Kepler motion.

Ad 1): According to the above arguments the following hydrogen solution of relativistic wave mechanics [5] (p. 529) has also to be the hydrogen solution of electrodynamics:

$$\Psi^{\mathrm{HYD}} = \Psi^{\mathrm{HYD,Re}} + \Psi^{\mathrm{HYD,Im}}$$

$$= \begin{cases} \Psi_1 = (l-m+1)\, R_1(r)\, P_{l+1}^{m}(\cos\vartheta)\, e^{im\varphi}\, e^{-i\omega t}, \\ \Psi_2 = R_1(r)\, P_{l+1}^{m+1}(\cos\vartheta)\, e^{i(m+1)\varphi}\, e^{-i\omega t}, \\ \Psi_3 = i(l+m+1)\, R_3(r)\, P_l^{m}(\cos\vartheta)\, e^{-im\varphi}\, e^{-i\omega t}, \\ \Psi_4 = -iR_3(r)\, P_l^{m+1}(\cos\vartheta)\, e^{i(m+1)\varphi}\, e^{-i\omega t} \end{cases} \quad (6)$$

with the characteristic frequencies

$$\nu = m_0 c^2/h \sqrt{1 + \frac{\alpha^2}{[n_r + \sqrt{(l+1)^2 - \alpha^2}]^2}}\,. \quad (7)$$

The hydrogen spinor (6), of whom only one of the two spin alternatives is reproduced here for the sake of simplicity, is a complex quantity and has four components. It decomposes into two wave fields, one real and the other imaginary. Since the four components of the spinor can not be related to space directions or otherwise be interpreted geometrically, both spinor fields lack any intuitive notion. According to current convention the expression (starred symbol = complex conjugate quantity)

$$\Psi^{\mathrm{HYD}} \cdot \overset{*}{\Psi}{}^{\mathrm{HYD}} = \sum_{k=1}^{4} [(\Psi_k^{\mathrm{Re}})^2 + (\Psi_k^{\mathrm{Im}})^2] \quad (8)$$

represents a probability for the location of the orbiting electron. "As strange as this point of view may be and as much difficulties in detail it even presents to its understanding, it is at the moment, however, the only way to unify particle- and wave interpretation; therefore we will follow is, without seeing more in it than a transition stage", [6] (p. 375).

If a field quantity propagates as a wave, all its components have the same phase velocity. A glance at (6) thus immediately shows that the hydrogen spinor field contains electromagnetic fields. For if one wanted to accept the hydrogen spinor in the form (6), one would be faced with a centrally symmetric wave field, whose second and fourth component – at fixed radius – would propagate with a different phase velocity than the first and third component. However, one easily arrives at a uniform wave process now by splitting off the factor $\exp(i\varphi) = \cos\varphi + i\sin\varphi$ from the second and fourth component and then proceeding according to [1], (7). With this one obtains the two photon fields of the electrodynamic hydrogen model, phase-synchronized in all components, of which, in agreement with [3], (4), one emerges as real ("Re-photon") and the other as imaginary ("Im-photon"):

$$\begin{aligned} &\Psi_1 = i(l-m+1)\, R_1 P_{l+1}^{m} e^{im\varphi} e^{-i\omega t} = iE_3, \\ &\Psi_2 = iR_1 P_{l+1}^{m+1} e^{im\varphi} e^{-i\omega t}(\cos\varphi + i\sin\varphi) = i(E_1 + iE_2), \\ &\Psi_3 = -(l+m+1)\, R_3 P_l^{m} e^{im\varphi} e^{-i\omega t} = H_3, \\ &\Psi_4 = R_3 P_l^{m+1} e^{im\varphi} e^{-i\omega t}(\cos\varphi + i\sin\varphi) = H_1 + iH_2, \end{aligned} \quad (9)$$

and

$$\begin{aligned} &E_1 = R_1 P_{l+1}^{m+1} e^{i(m\varphi-\omega t)} \cos\varphi = E_1^{\mathrm{Re}} + iE_1^{\mathrm{Im}}, \\ &E_2 = R_1 P_{l+1}^{m+1} e^{i(m\varphi-\omega t)} \sin\varphi = E_2^{\mathrm{Re}} + iE_2^{\mathrm{Im}}, \\ &E_3 = R_1 P_{l+1}^{m} e^{i(m\varphi-\omega t)}(l-m+1) = E_3^{\mathrm{Re}} + iE_3^{\mathrm{Im}}, \\ &H_1 = R_3 P_l^{m+1} e^{i(m\varphi-\omega t)} \cos\varphi = H_1^{\mathrm{Re}} + iH_1^{\mathrm{Im}}, \end{aligned} \quad (10)$$

$$H_2 = R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \sin\varphi = H_2^{\mathrm{Re}} + i\, H_2^{\mathrm{Im}},$$
$$H_3 = -R_3\, P_l^m\, e^{i(m\varphi-\omega t)}\,(l+m+1) = H_3^{\mathrm{Re}} + i\, H_3^{\mathrm{Im}}.$$

The additional factors (9) appear in the second and fourth spinor component of Dirac's hydrogen model (6), i.e. exactly where they are needed because of [1], (7). This shows clearly that the wave-mechanical hydrogen spinor is, in a way, prepared for its further transfer into the electromagnetic hydrogen model (11).

The subsequent treatment of the two centrally symmetric photon fields (11) is guided by the condition of perpendicularity [7], (7). This condition requires, together with [1], (12), that all field vectors of the two photons have to be perpendicular to all radial directions. Or: the field vectors are tangent to spheres around the origin. Thus the condition requires the pure transversality of the two photon fields and will be called "transversality condition" in the following. It goes beyond the apparatus of relativistic wave mechanics and represents the most important aid in constructing solutions in electromagnetic hydrogen theory.

If the two photon fields in (11) are viewed as standing waves, then they oscillate synchronized and in the same phase for all components. This would lead to an exclusively linear polarization, in contrast to experience. Therefore one firstly has to intervene – at best already in (10) – in such a way, that elliptical polarizations arise in (11). Let us illustrate this process for example by considering Ψ_4 of (9):

$$\Psi_4 = R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \cos\varphi + i\, R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \sin\varphi = H_1 + i\, H_2 \tag{11}$$

with the obvious assumption

$$H_1 = R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \cos\varphi \quad \text{and}$$
$$H_2 = R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \sin\varphi. \tag{12}$$

Equations (12), with the help of the substitution

$$H_1 = A\cos\varphi, \quad H_2 = A\sin\varphi,$$
$$A = R_3\, P_l^{m+1}\, e^{im\varphi}\, e^{-i\omega t}, \tag{13}$$

read in spherical coordinates, show that the partial vector parallel to the 1–2 plane,

$$\boldsymbol{\Psi}_4 = \boldsymbol{H}_1 + \boldsymbol{H}_2, \tag{14}$$

oscillates only in radial direction, whereas for the required elliptical polarization of $\boldsymbol{H}$ it should at least be circularly polarized. We therefore rewrite Ψ_4 as follows:

$$\Psi_4 = \tfrac{1}{2}\Psi_4 + i \exp\{-i\,\pi/2\}\,\tfrac{1}{2}\Psi_4 = H_1 + i\, H_2, \tag{15}$$

or in detail with (12):

$$\begin{aligned}\Psi_4 = {} & \tfrac{1}{2}\,(R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \cos\varphi \\ & + i\, R_3\, P_l^{m+1}\, e^{i(m\varphi-\omega t)} \sin\varphi) \\ & + \tfrac{1}{2}[i\, R_3\, P_l^{m+1} \exp\{i\,[m\varphi-(\omega t+\pi/2)]\} \cos\varphi \\ & - R_3\, P_l^{m+1} \exp\{i\,[m\varphi-(\omega t+\pi/2)]\} \sin\varphi] \\ = {} & H_1 + i\, H_2,\end{aligned} \tag{16}$$

and

$$\begin{aligned}H_1 = {} & \tfrac{1}{2}\, R_3\, P_l^{m+1}\,(e^{i(m\varphi-\omega t)} \cos\varphi \\ & - \exp\{i\,[m\varphi-(\omega t+\pi/2)]\} \sin\varphi), \\ H_2 = {} & \tfrac{1}{2}\, R_3\, P_l^{m+1}\,(e^{i(m\varphi-\omega t)} \sin\varphi \\ & + \exp\{i\,[m\varphi-(\omega t+\pi/2)]\} \cos\varphi).\end{aligned} \tag{17}$$

In (16) the term in round brackets, interpreted as partial vector $\boldsymbol{\Psi}_4$, would oscillate radially, the other term, however, tangentially with a phase shift of $\pi/2$. Both together yield a partial vector (15) possessing the required circular polarization. Since according to (10) the first spinor component

$$\Psi_3 = -(l+m+1)\, R_3\, P_l^m\, e^{i(m\varphi-\omega t)} = H_3 \tag{18}$$

eventually oscillates synchronized to the term of (16) in round brackets, the desired elliptical polarization of $\boldsymbol{H}$ results, which should be perpendicular to the radial directions.

In anticipation it should be said that the desired transversality has not yet been realized here at all, since the wave mechanical model (6) represents a largely inadequate solution with regard to the field structure. Obviously a substantial structural shortcoming is caused by the very fact that the half-integer spin does not manifest itself numerically. The quantum numbers l and m enter as integers here. In the literature l is usually rewritten ad hoc as a half-integer quantum number [5] (p. 530), [6] (p. 488), where simply the lower indices of the Legendre functions in (6) are averaged over. One has never tried to remove this deficiency since the exact field structure is irrelevant for the probabilistic interpretation. And that, although the probability interpretation encounters formidable difficulties especially in the case of the hydrogen field. Schrödinger: "In that case the orbiting electron would be positioned preferentially at the origin, since there the probability for its location becomes

infinite. This, however, is exactly the place where it never can be because there the nucleus is located. If one tries to avoid this difficulty by setting the nucleus into motion too, then the relevant amplitude equation is replaced by the equation

$$\left[\Delta + \frac{8\pi^2}{h^2}\,\frac{m_0 M}{m_0 + M}\,(U - \Phi)\right]\Psi = 0, \qquad (19)$$

which, due to the symmetry of the reduced mass, can not any more give information about whether the solution for the probability field is to be attributed to the electron m or the nucleus M.

Together with (17), (18), and an analogous computation for the electric vector, we obtain from (6) in summary the following hydrogen field:

$$\begin{aligned}
E_1 &= \tfrac{1}{2} R_1 P_{l+1}^{m+1} (e^{i(m\varphi-\omega t)} \cos\varphi \\
&\quad - \exp\{i[m\varphi-(\omega t+\pi/2)]\} \sin\varphi) = E_1^{\mathrm{Re}} + i E_1^{\mathrm{Im}}, \\
E_2 &= \tfrac{1}{2} R_1 P_{l+1}^{m+1} (e^{i(m\varphi-\omega t)} \sin\varphi \\
&\quad + \exp\{i[m\varphi-(\omega t+\pi/2)]\} \cos\varphi) = E_2^{\mathrm{Re}} + i E_2^{\mathrm{Im}}, \\
E_3 &= (l-m+1)\, R_1 P_{l+1}^{m}\, e^{i(m\varphi-\omega t)} = E_3^{\mathrm{Re}} + i E_3^{\mathrm{Im}}, \\
H_1 &= \tfrac{1}{2} R_3 P_l^{m+1} (e^{i(m\varphi-\omega t)} \cos\varphi \\
&\quad - \exp\{i[m\varphi-(\omega t+\pi/2)]\} \sin\varphi) = H_1^{\mathrm{Re}} + i H_1^{\mathrm{Im}}, \\
H_2 &= \tfrac{1}{2} R_3 P_l^{m+1} (e^{i(m\varphi-\omega t)} \sin\varphi \\
&\quad + \exp\{i[m\varphi-(\omega t+\pi/2)]\} \cos\varphi) = H_2^{\mathrm{Re}} + i H_2^{\mathrm{Im}}, \\
H_3 &= -(l+m+1)\, R_3 P_l^{m}\, e^{i(m\varphi-\omega t)} = H_3^{\mathrm{Re}} + i H_3^{\mathrm{Im}} \qquad (20)
\end{aligned}$$

oscillating with the characteristic frequencies (7). In the first instance we convince ourselves that (19) doesn't represent a transverse solution yet by taking the inner product of the electric field of the Re-photon with the radial unit vector in order to observe that this product doesn't vanish:

$$\begin{aligned}
\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{r}^0 = \{ & \tfrac{1}{2} R_1 P_{l+1}^{m+1} \{\cos(m\varphi - \omega t)\cos\varphi \\
& - \cos[m\varphi - (\omega t + \tfrac{\pi}{2})]\sin\varphi\}, \\
& \tfrac{1}{2} R_1 P_{l+1}^{m+1} \{\cos(m\varphi - \omega t)\sin\varphi \\
& + \cos[m\varphi - (\omega t + \tfrac{\pi}{2})]\cos\varphi\}, \qquad (21) \\
& (l-m+1)\, R_1 P_{l+1}^{m} \cos(m\varphi - \omega t)\} \\
& \cdot (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \neq 0
\end{aligned}$$

or

$$\begin{aligned}
\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{r}^0 = \{ & \tfrac{1}{2} R_1 P_{l+1}^{m+1} \cos[(m+1)\varphi - \omega t], \\
& \tfrac{1}{2} R_1 P_{l+1}^{m+1} \sin[(m+1)\varphi - \omega t],
\end{aligned}$$

$$\begin{aligned}
& (l-m+1)\, R_1 P_{l+1}^{m} \cos(m\varphi - \omega t)\} \\
& \cdot (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \neq 0. \qquad (22)
\end{aligned}$$

The results (22) may be immediately verified e.g. for small l and m.

A further substantial deficiency is evident directly from (22): Despite the splitting off of the factors $\exp(i\varphi)$ the first and second vector components of $\boldsymbol{E}^{\mathrm{Re}}$ possess different phase velocities than the third because of the manipulation (5).

The wave described by (22), which runs around the 3-axis, is obviously not of the kind of a light-wave in the usual sense. For the ray vector, because of the transversality condition, has to point everywhere, where it doesn't vanish, into the radial direction, and thus the wave propagates perpendicular to the ray vector everywhere. For large radii this seems to imply that the electromagnetic fields of the two hydrogen-photons may be shifted perpendicular to the ray direction as quickly as one liked. On the other hand, however, the fields may be visualized as standing waves. In anticipation it should be mentioned that the orbiting waves will turn out as the electrodynamic manifestations of angular momentum and spin of the hydrogen field.

The inadequate model (20), on the other hand, reveals already several essential structural properties of the hydrogen field.

One immediately sees that everywhere and at all times the following relations hold:

$$\boldsymbol{E}^{\mathrm{Re}} \perp \boldsymbol{E}^{\mathrm{Im}} \quad \text{and} \quad \boldsymbol{H}^{\mathrm{Re}} \perp \boldsymbol{H}^{\mathrm{Im}}, \qquad (23)$$

or

$$\boldsymbol{E}^{\mathrm{Re}} \cdot \boldsymbol{E}^{\mathrm{Im}} = 0 \quad \text{and} \quad \boldsymbol{H}^{\mathrm{Re}} \cdot \boldsymbol{H}^{\mathrm{Im}} = 0. \qquad (24)$$

This means that the two hydrogen-photons are perpendicular to each other everywhere and at all times or, which is the same, they are mutually orthogonal pointwise. Furthermore one sees directly that for the case of a transverse model the electric and magnetic field vectors in both photon fields are parallel or antiparallel, so that the radiation vanishes everywhere and at all times:

$$\begin{aligned}
S^{\mathrm{Re}} &= \frac{c}{4\pi}\, \boldsymbol{E}^{\mathrm{Re}} \times \boldsymbol{H}^{\mathrm{Re}} = 0 \quad \text{and} \\
S^{\mathrm{Im}} &= \frac{c}{4\pi}\, \boldsymbol{E}^{\mathrm{Im}} \times \boldsymbol{H}^{\mathrm{Im}} = 0. \qquad (25)
\end{aligned}$$

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